AFAD 2026

Asian Forum for Accelerators and Detectors

13-17 October 2026 Shanghai, China



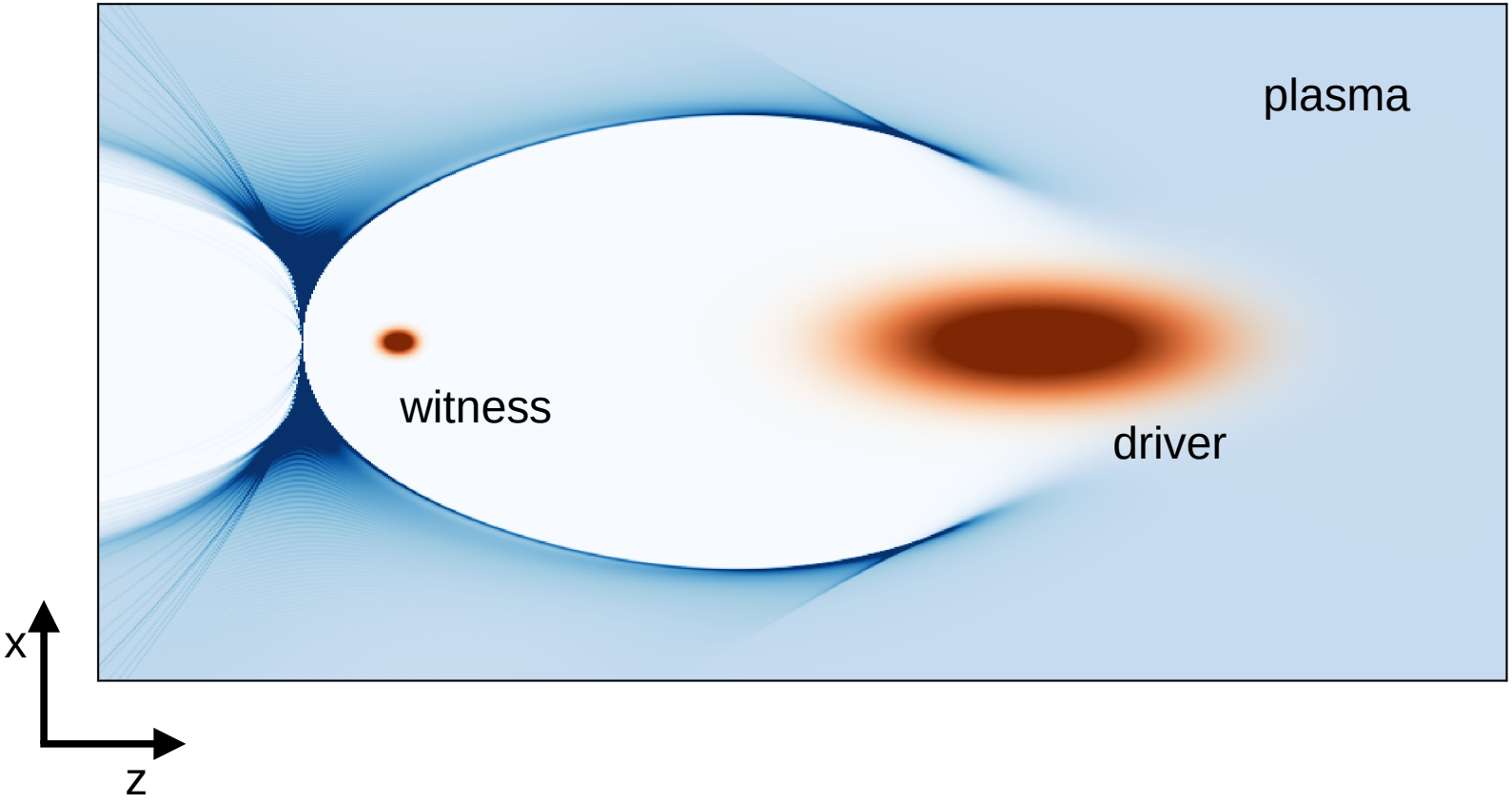
# Accuracy analysis of particle pushers for PIC simulations


Petr Tuev

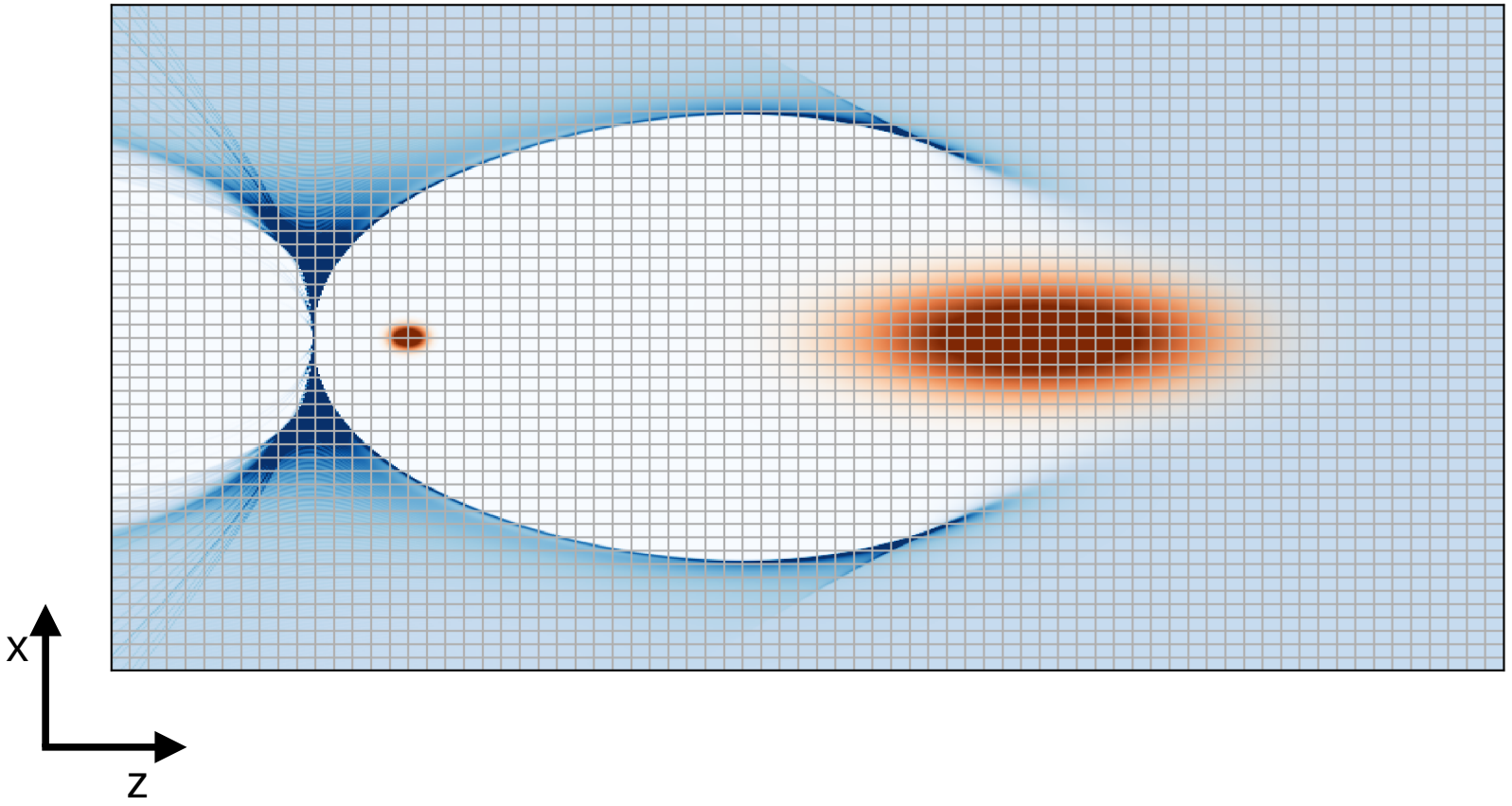
Budker Institute of Nuclear Physics SB RAS, Novosibirsk, Russia

Novosibirsk State University, Novosibirsk, Russia

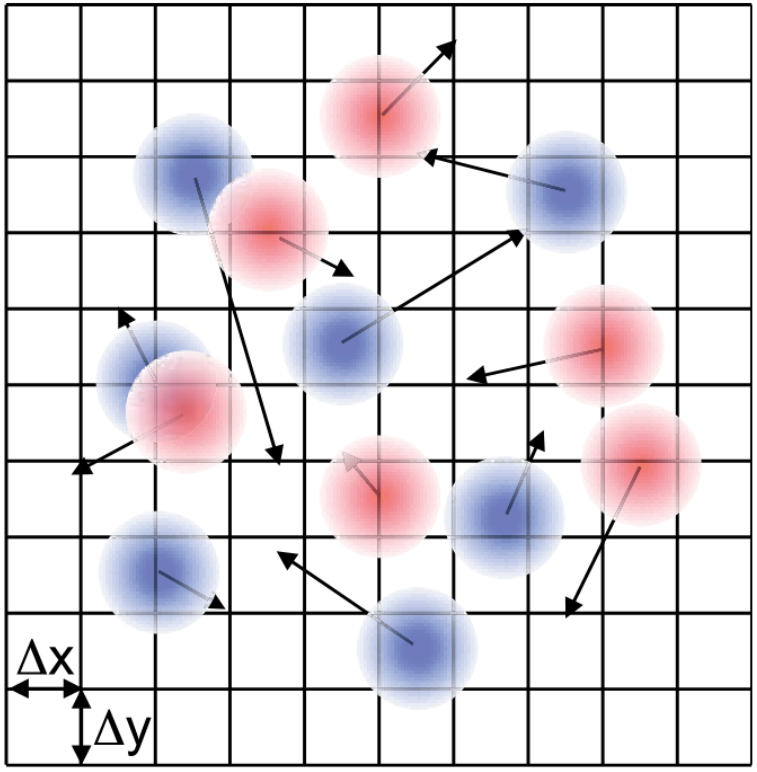
# Particle-in-cell method



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# Particle-in-cell method




$$Maxwell's\ equations$$

$$rot\ \boldsymbol{E} = -\frac{d\,\boldsymbol{B}}{dt}$$

$$rot\ \boldsymbol{B} = \frac{d\,\boldsymbol{E}}{dt} + \boldsymbol{j}$$

$$Equations\ of\ motion$$

$$\frac{d\,\boldsymbol{r}}{dt} = \boldsymbol{v}$$

$$\frac{d\,\boldsymbol{u}}{dt} = \frac{q}{m}\left(\boldsymbol{E} + [\,\boldsymbol{v}_{aver} \times \boldsymbol{B}\,]\right)$$

$$\boldsymbol{r} = (x, y, z), \boldsymbol{u} = \gamma\,\boldsymbol{v}, \boldsymbol{v} - velocity$$

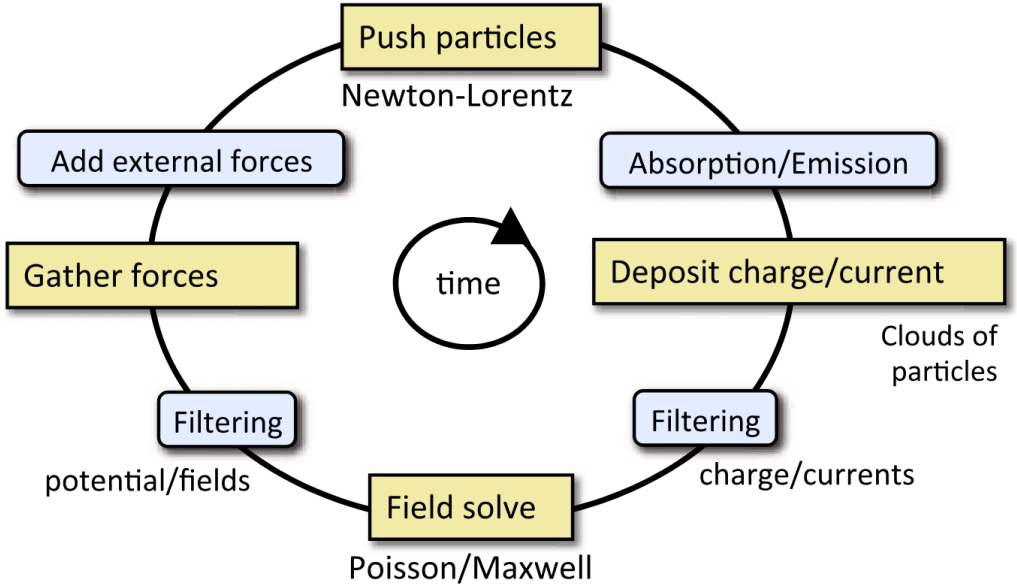
$$1/\gamma = \sqrt{1 - (\boldsymbol{v} \cdot \boldsymbol{v})}$$

$$\boldsymbol{E}/\boldsymbol{B} - electric/magnetic\ fields$$

$$Dimensioning: c = e = m_e = n_0 = 1$$

J.-L. Vay and R. Lehe, Rev. Accl. Sci. Tech., vol. 9, pp.165–186, 2016

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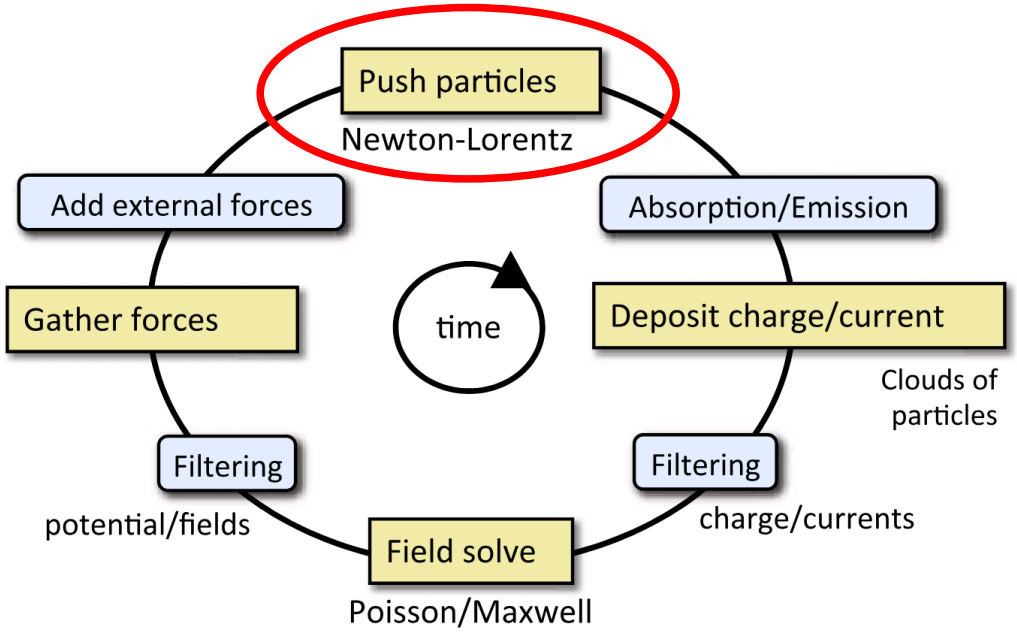
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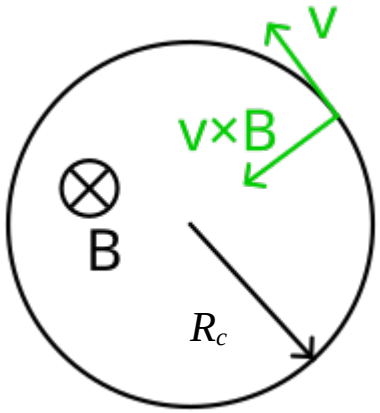
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J.-L. Vay and R. Lehe, Rev. Accl. Sci. Tech., vol. 9, pp.165–186, 2016

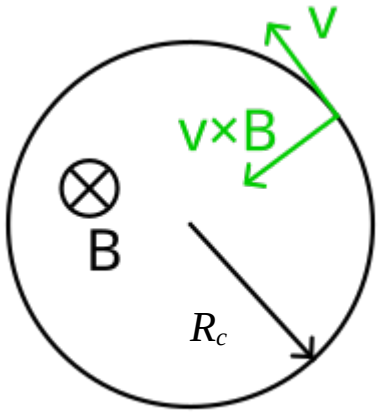
# Why should we care about this?

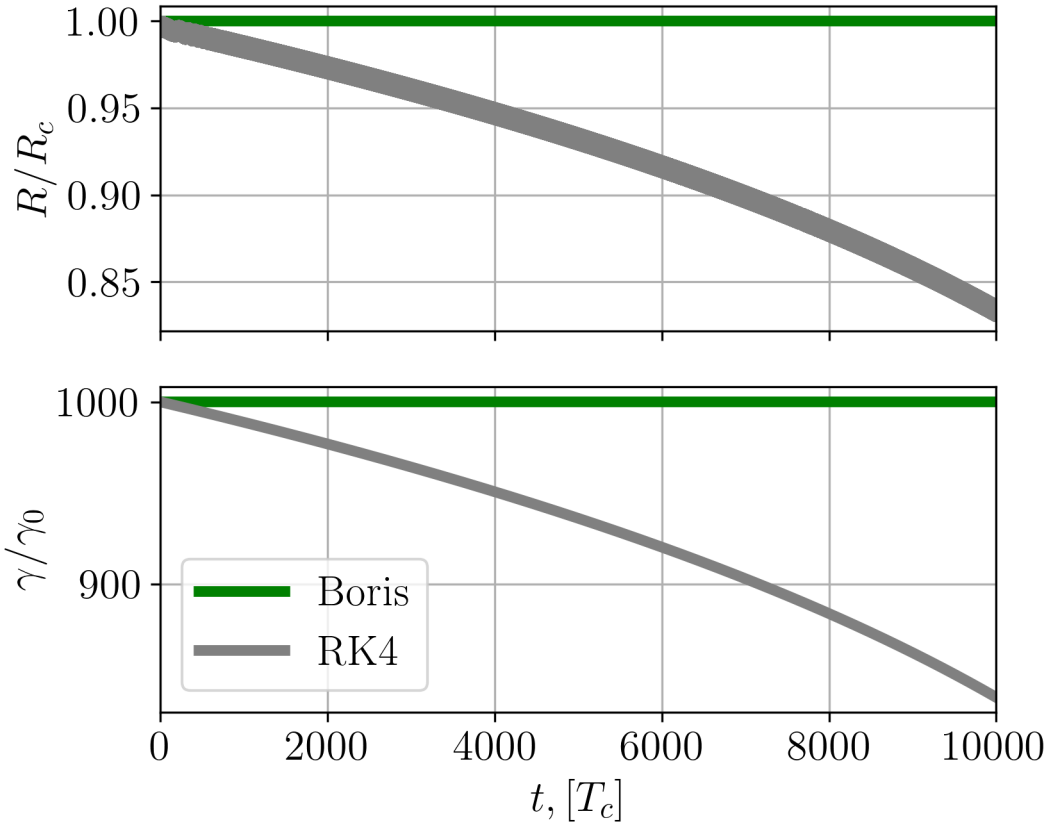
Cyclotron motion



# Why should we care about this?

Cyclotron motion

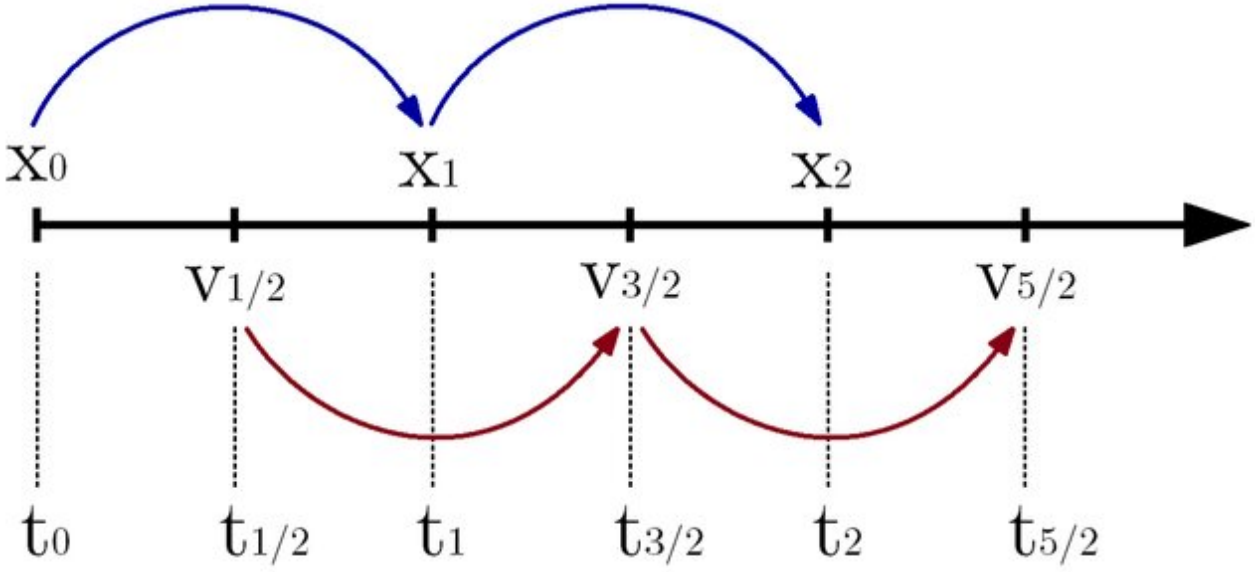


# Particle pushers

$$\frac{\boldsymbol{r}^{n}-\boldsymbol{r}^{n+1}}{\Delta t}=\boldsymbol{v}^{n+1/2}$$

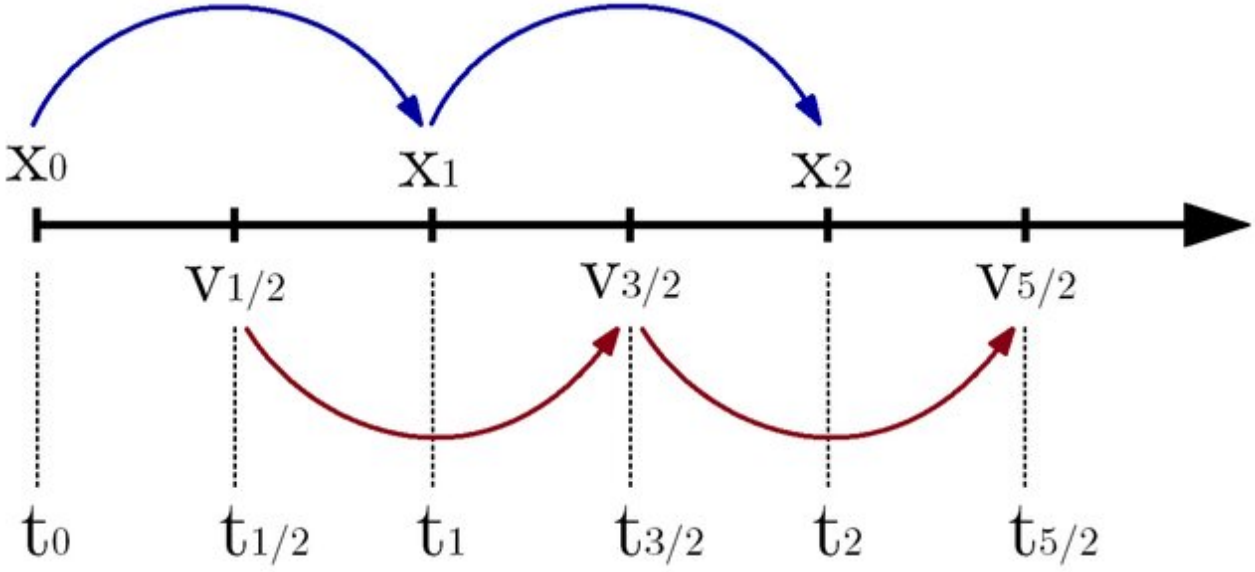
$$\frac{\boldsymbol{u}^{n-1/2}-\boldsymbol{u}^{n+1/2}}{\Delta t}=\frac{q}{m}\left(\boldsymbol{E}^{n}+\left[\boldsymbol{v}_{aver}\times\boldsymbol{B}^{n}\right]\right)$$



# Particle pushers

$$\frac{\boldsymbol{r}^{n}-\boldsymbol{r}^{n+1}}{\Delta t}=\boldsymbol{v}^{n+1/2}$$

$$\frac{\boldsymbol{u}^{n-1/2}-\boldsymbol{u}^{n+1/2}}{\Delta t}=\frac{q}{m}\left(\boldsymbol{E}^{n}+\left[\boldsymbol{v}_{aver}\times\boldsymbol{B}^{n}\right]\right)$$



# Particle pushers

$$\boldsymbol{v}_{aver} = \frac{\boldsymbol{u}^{n} + \boldsymbol{u}^{n+1}}{2\gamma_{aver}}$$

## Boris-tg

$$\gamma_{aver} = \gamma_{-} = \gamma_{+}$$

$$(1)\quad \boldsymbol{u}_{-} = \boldsymbol{u}^{n-1/2} + (q\boldsymbol{E}^{n}/m)\cdot(\Delta t/2)$$

$$(2)\quad \gamma_{aver} = \sqrt{1 + (\boldsymbol{u}_{-}\cdot\boldsymbol{u}_{-})}$$

$$(3)\quad \boldsymbol{t} = \tan\left(qB_0^n \Delta t/(2m\gamma_{aver})\right)\boldsymbol{B}^{n}/B_0^n$$

$$(4)\quad \boldsymbol{s} = 2\boldsymbol{t}/(1 + (\boldsymbol{t}\cdot\boldsymbol{t}))$$

$$(5)\quad \tilde{\boldsymbol{u}} = \boldsymbol{u}_{-} + \boldsymbol{u}_{-}\times\boldsymbol{t}$$

$$(6)\quad \boldsymbol{u}_{+} = \boldsymbol{u}_{-} + \tilde{\boldsymbol{u}}\times\boldsymbol{s}$$

$$(7)\quad \boldsymbol{u}^{n+1/2} = \boldsymbol{u}_{+} + (q\boldsymbol{E}^{n}/m)\cdot(\Delta t/2)$$

J. P. Boris. // Proceeding of Fourth Conference on Numerical Simulations of Plasmas. – 1970. – P. 3–67.

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J. P. Boris. // Proceeding of Fourth Conference on Numerical Simulations of Plasmas. – 1970. – P. 3–67.

## Higuera–Cary

$$\gamma_{aver} = \sqrt{\left(1 + (\boldsymbol{u}^{n} + \boldsymbol{u}^{n+1})^{2}/4\right)}$$

A. V. Higuera and J. R. Cary. // Physics of plasmas. – 2017. – Vol. 24. – P. 052104.

## Lapenta-Markidis

$$2\gamma_{aver} = \gamma^{n} + \gamma^{n+1}$$

G. Lapenta, and S. Markidis. // Physics of Plasmas. - 2011. - Vol. 18. - P. 072101.
F. Bacchini // The Astrophysical Journal Supplement Series. - 2023. - Vol. 268. - P. 60

# Particle pushers

## Vay

$$\boldsymbol{v}_{aver} = \frac{\boldsymbol{u}^{n}}{2\gamma^{n}} + \frac{\boldsymbol{u}^{n+1}}{2\gamma^{n+1}}$$

$$(1)\quad \boldsymbol{u}^{n} = \boldsymbol{u}^{n-1/2} + \frac{q\Delta t}{2m}\left(\boldsymbol{E}^{n} + \left[\frac{\boldsymbol{u}^{n-1/2}}{\gamma^{n-1/2}} \times \boldsymbol{B}^{n}\right]\right)$$

$$(2)\ \ldots\ (8)$$

$$(9)\quad \boldsymbol{u}^{n+1/2} = s\left[\boldsymbol{u}' + (\boldsymbol{u}'\cdot\boldsymbol{t})\boldsymbol{t} + \boldsymbol{u}' \times \boldsymbol{t}\right]$$

J.-L. Vay. // Physics of plasmas. – 2008. – Vol. 15. – P. 056701.

# Particle pushers

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J.-L. Vay. // Physics of plasmas. – 2008. – Vol. 15. – P. 056701.

Based on analytical solution for [**E**x**B**] drift and spliting of motion along and perpendicular to **B**

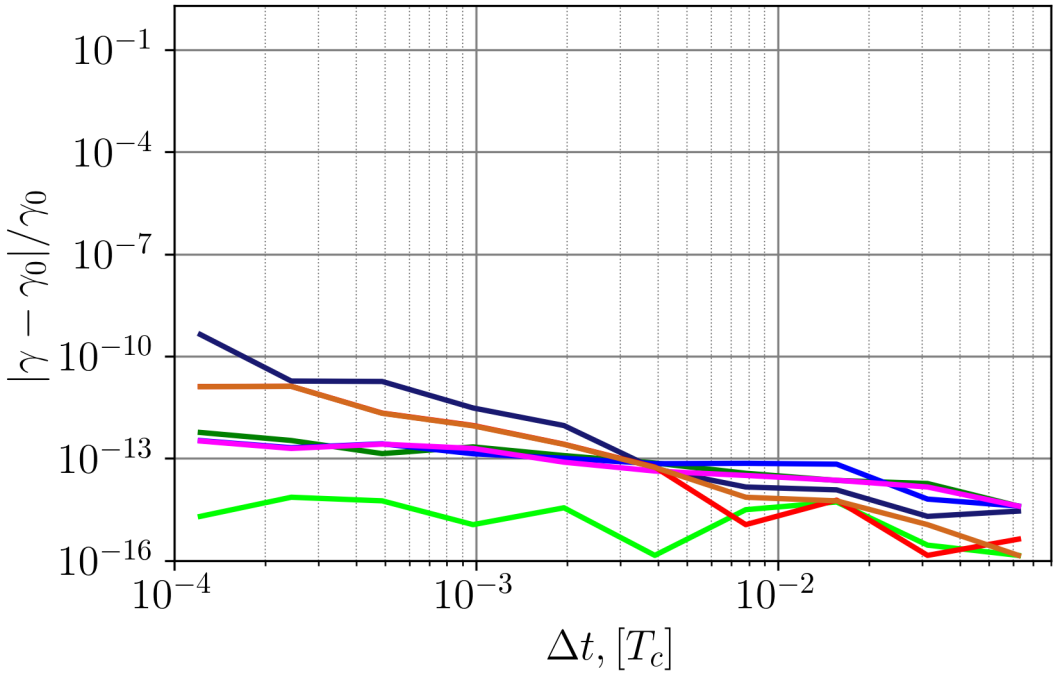
## VD1

K. V. Vshivkov, E. S. Voropaeva, A. A. Efimova. // Computational Technologie. – 2023. – Vol. 28. – P. 27.

## Umeda

T. Umeda. // Journal of Computational Physics. – 2023. – Vol. 472. – P. 111694.
T. Umeda and R. Ozaki. // Earth, Planets and Space. – 2025. – Vol. 75. – P. 157.

# Cyclotron motion




$B_z = 1$

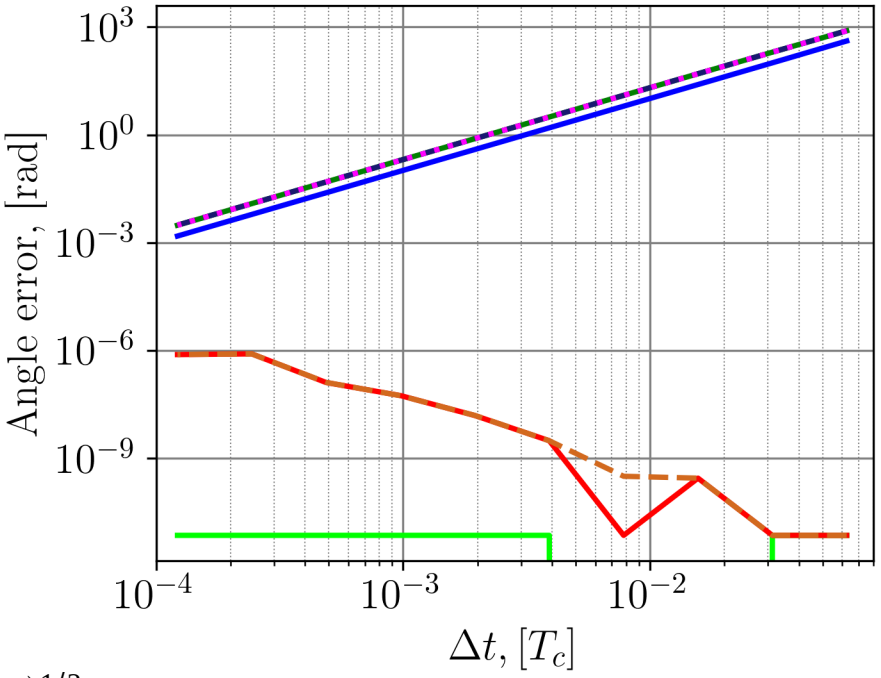
$\gamma = 100$

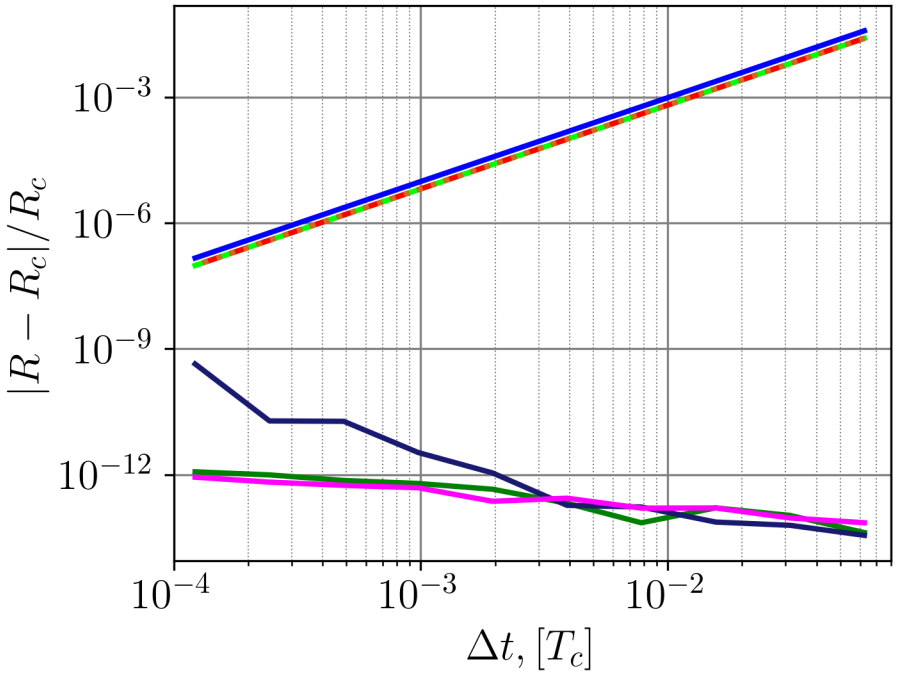
$u_y = -(\gamma^2 - 1)^{1/2}$

$t_{max} = 10^4 T_c$

Boris Boris-tg Vay HC LM VD1 Umeda

# Cyclotron motion




$B_z = 1$

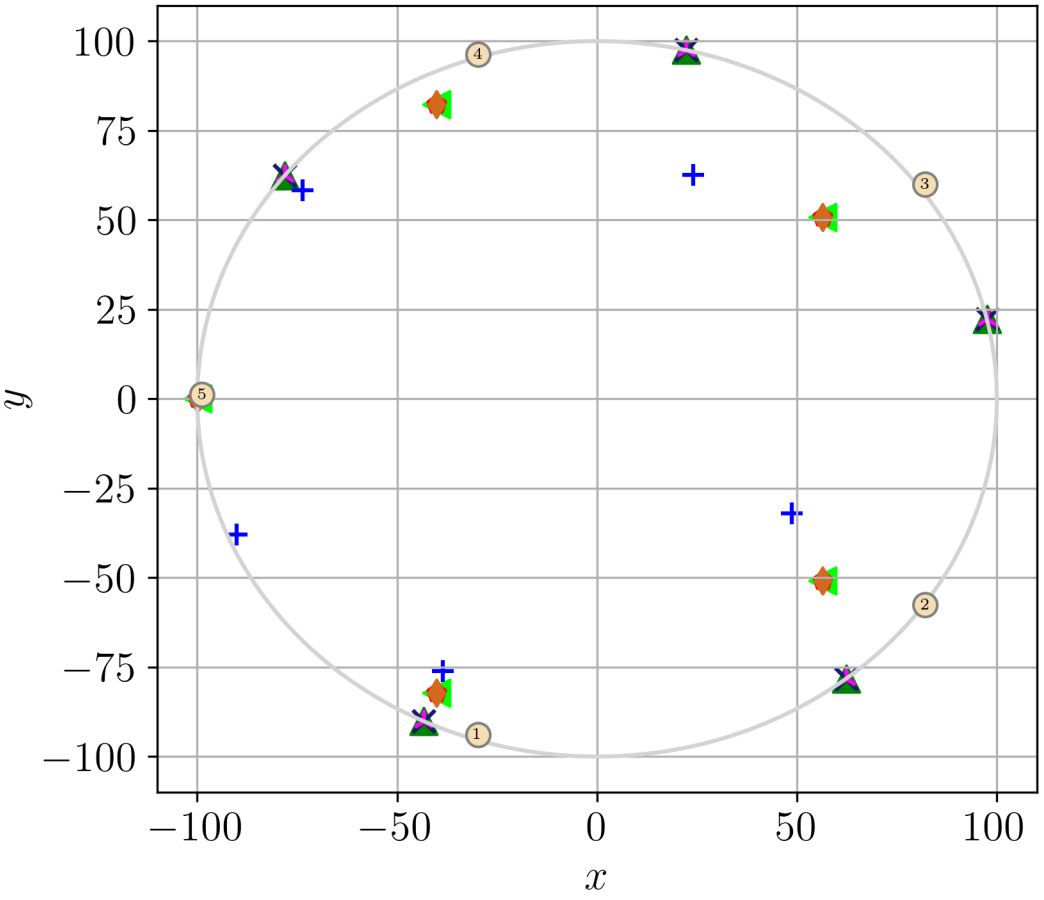
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Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

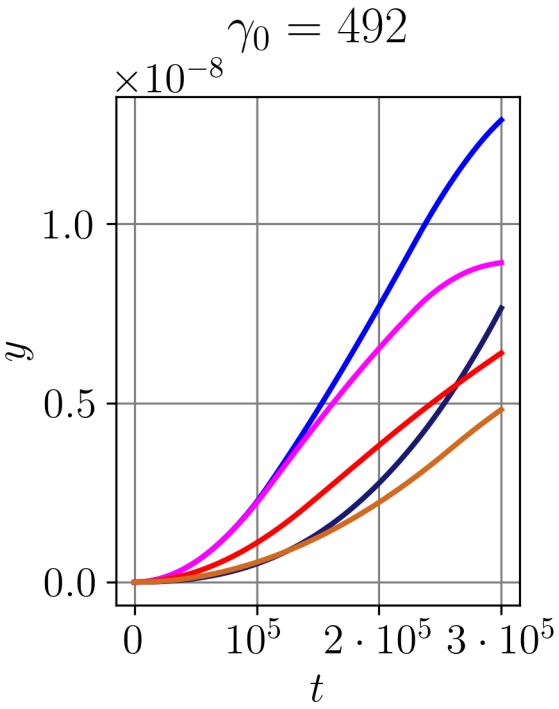
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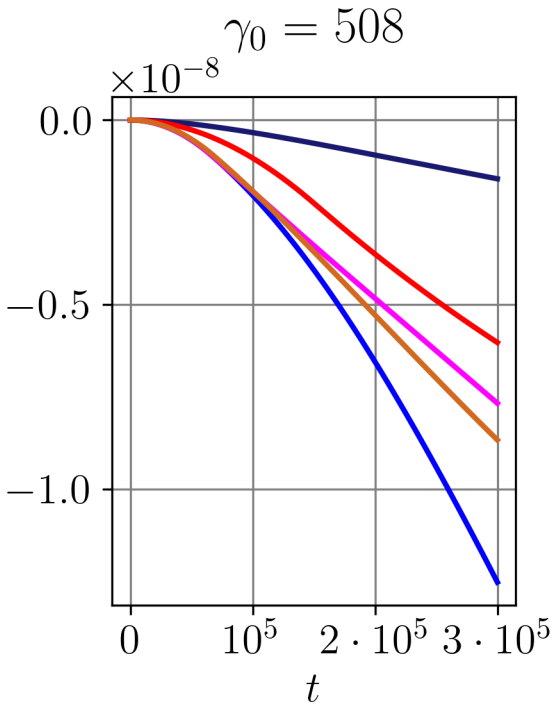


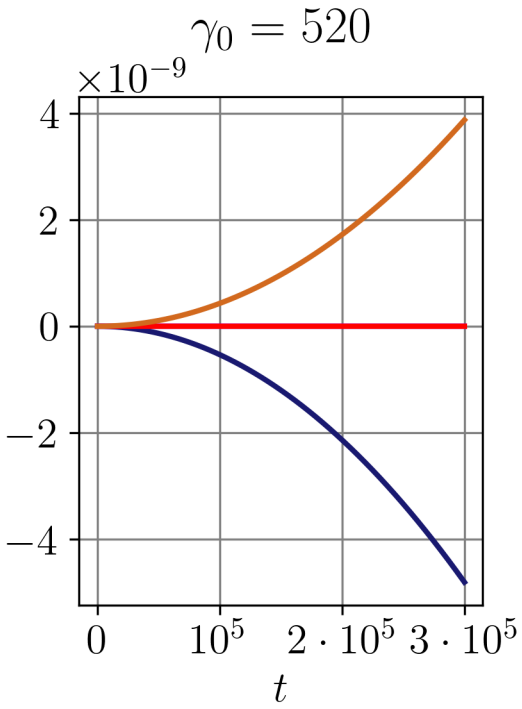

In this test, the pushers were divided into two groups. The first group includes Boris-tg, VD1, and Umeda, which calculate the rotation angle more accurately. Boris, Vay, and LM calculate the radius of rotation more accurately. The HC integrator falls somewhere in between.

Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

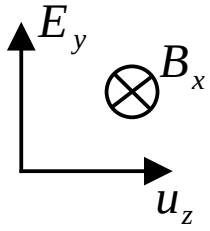
# Force-free configuration




$F_y = 0$




$B_x = 1$

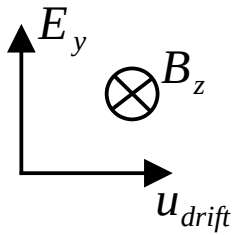
$u_z = (\gamma_0^2 - 1)^{1/2}$

$E_y = -v_z B_x$

For Boris and Boris-tg, the error is higher by approximately 10 orders of magnitude.

Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

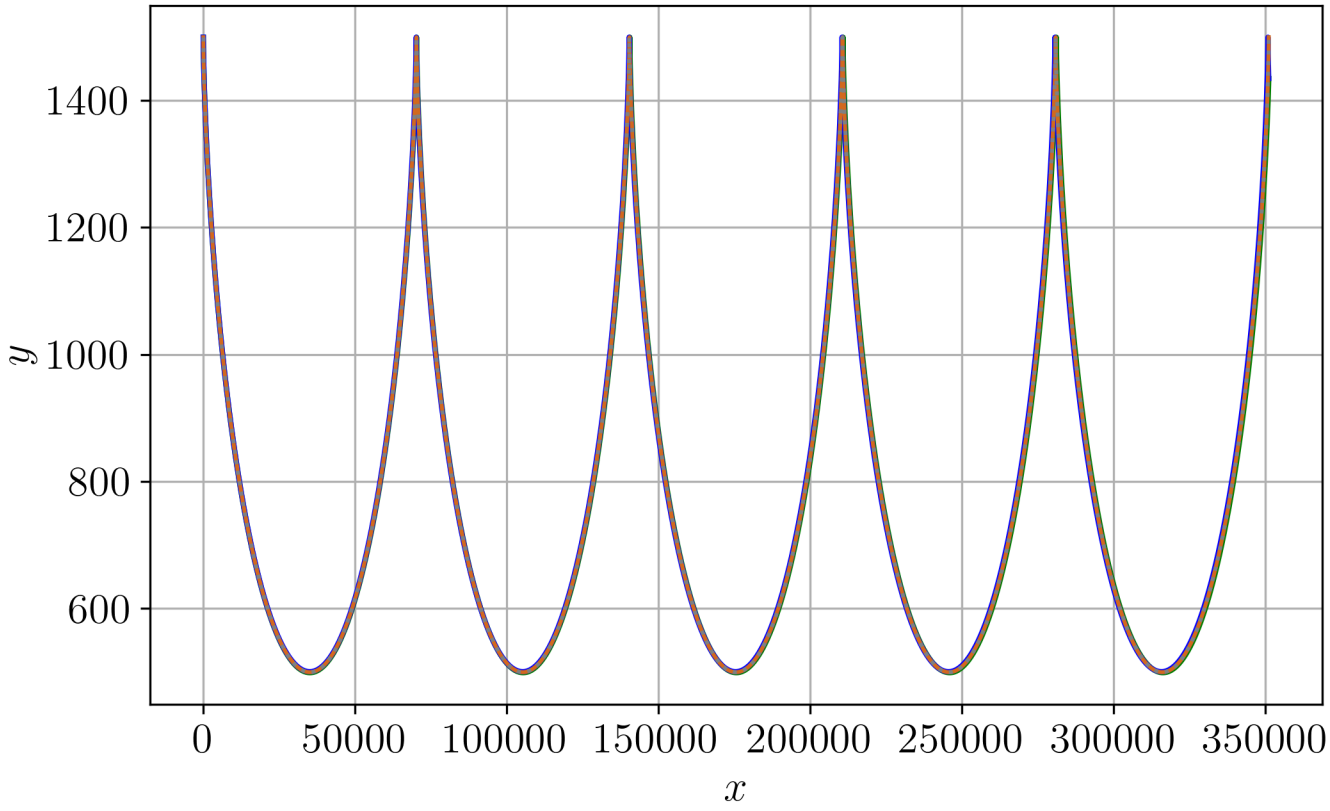
# E-cross-B drift




$B_z = 1$

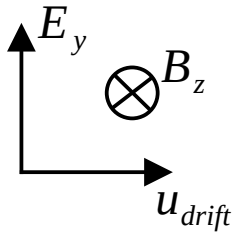
$E_y = B_z - 10^{-3}$

$\gamma_0 = 1$




Boris Boris-tg Vay HC LM VD1 Umeda

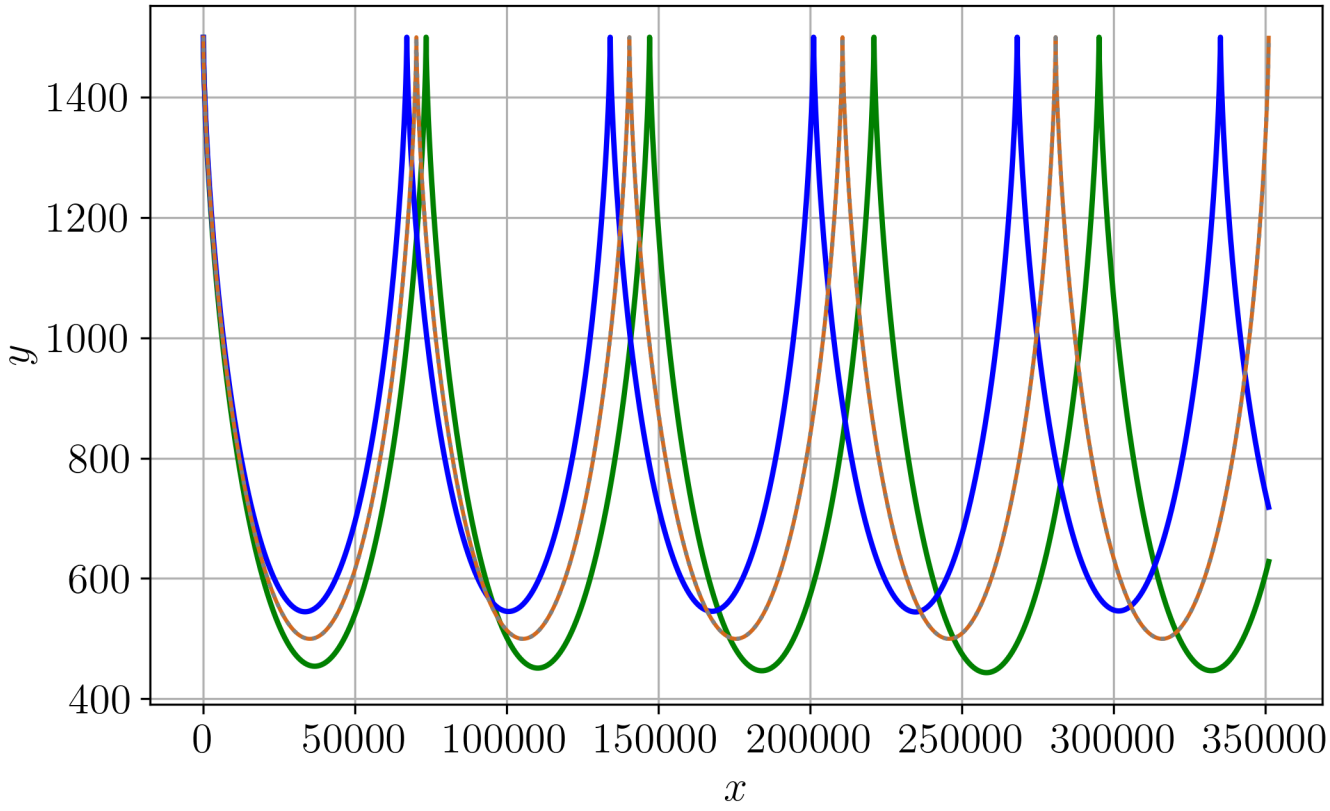
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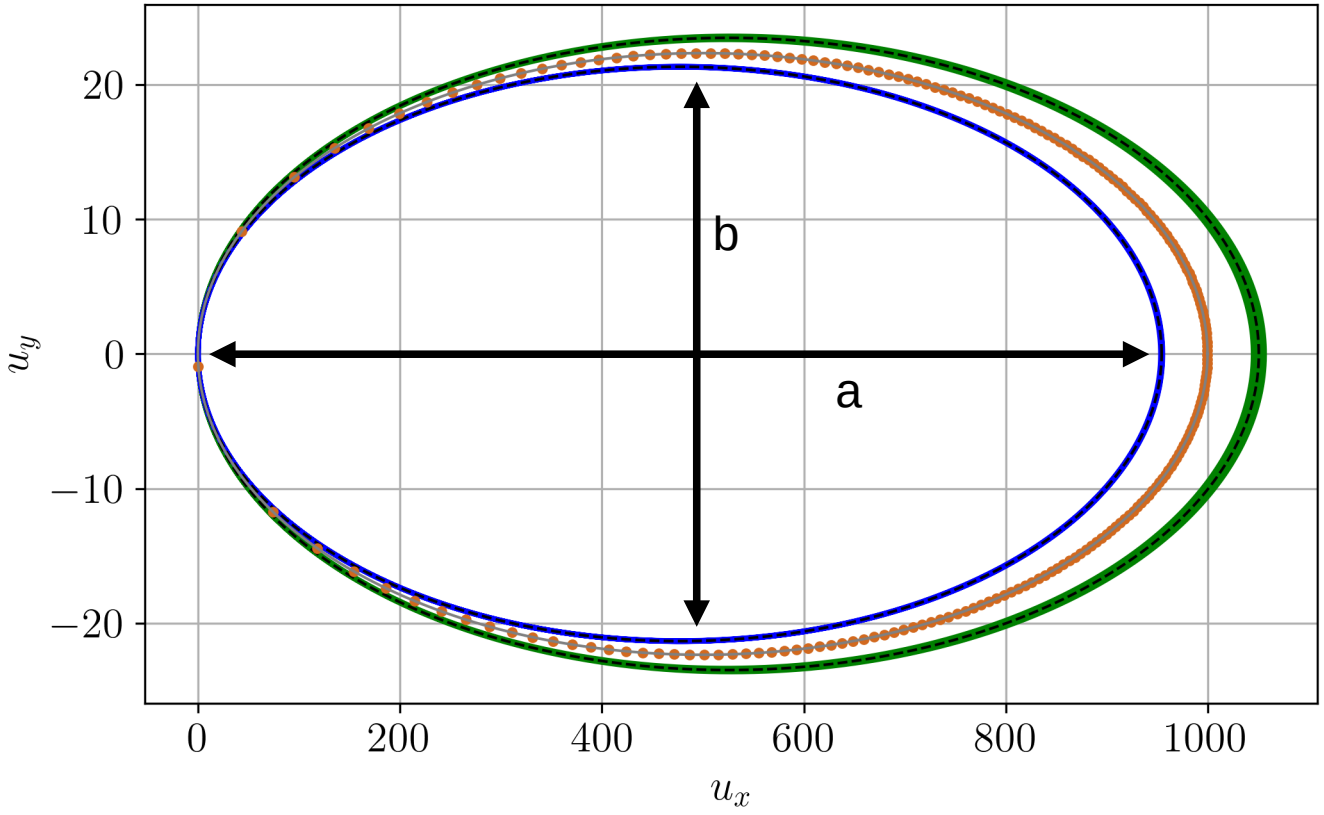
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Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

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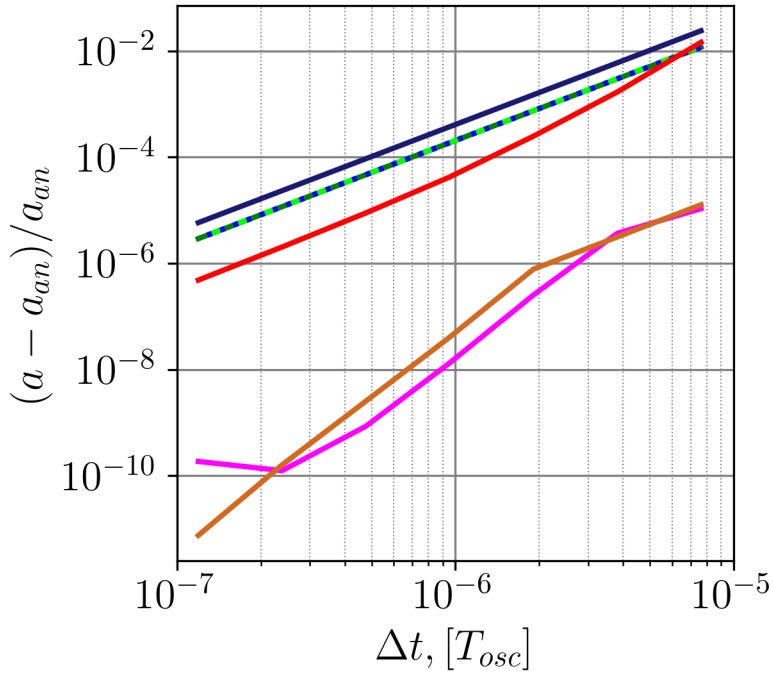

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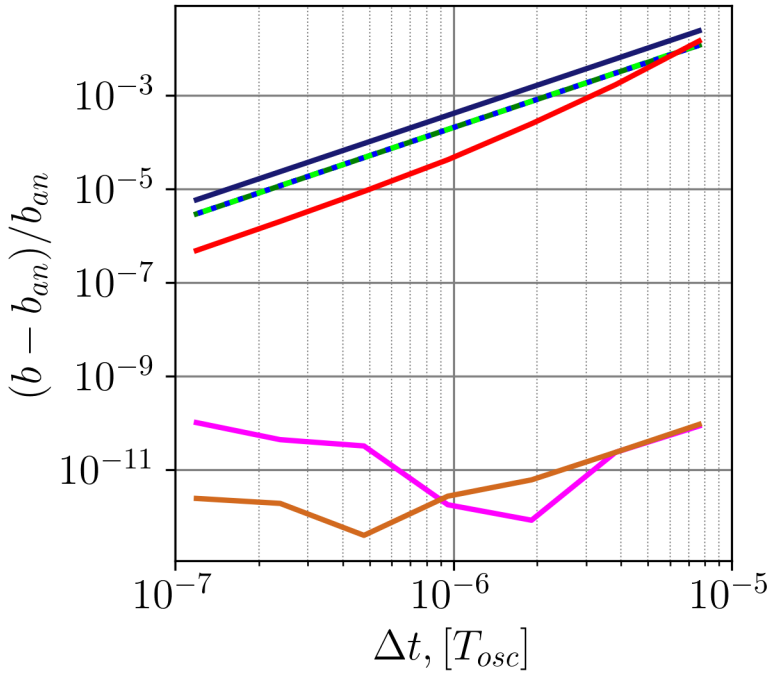
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Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

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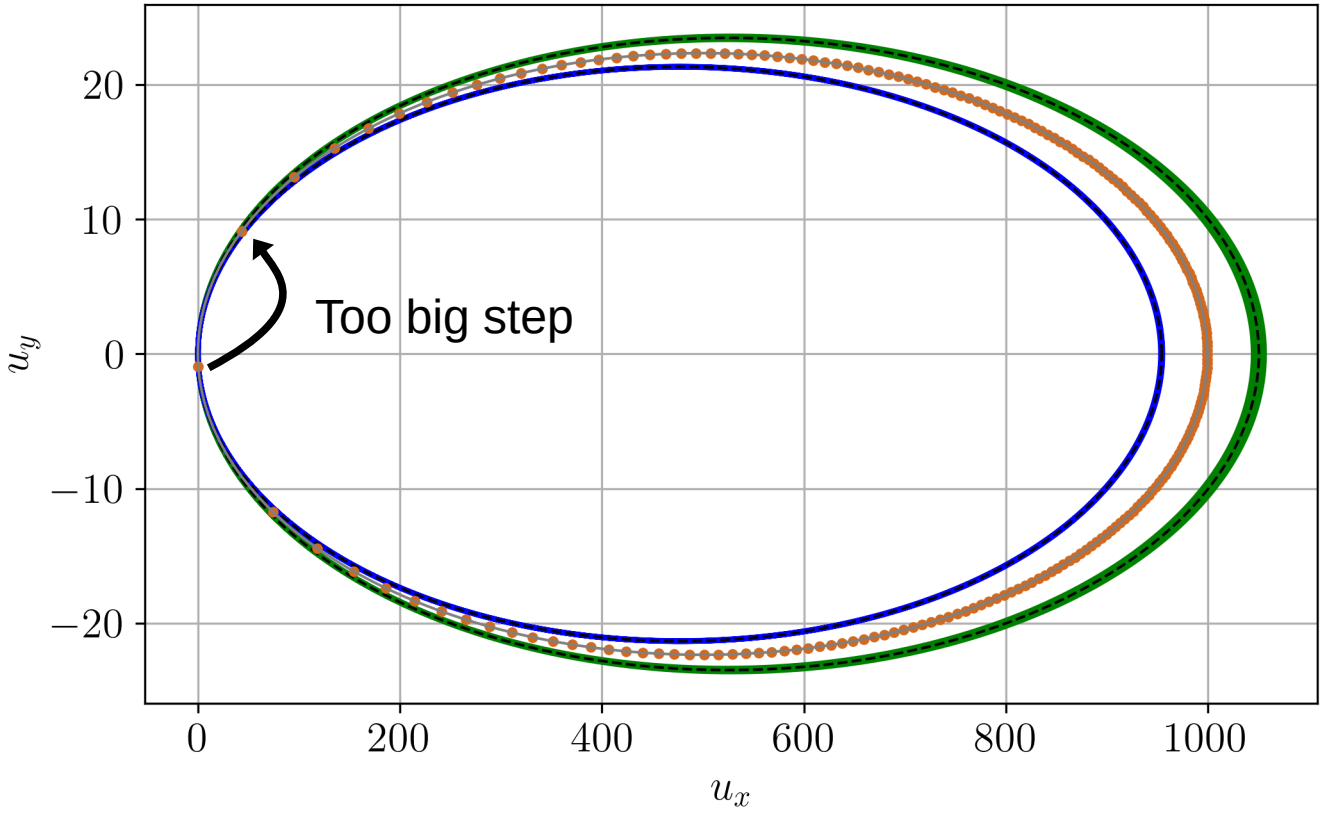

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# E-cross-B drift



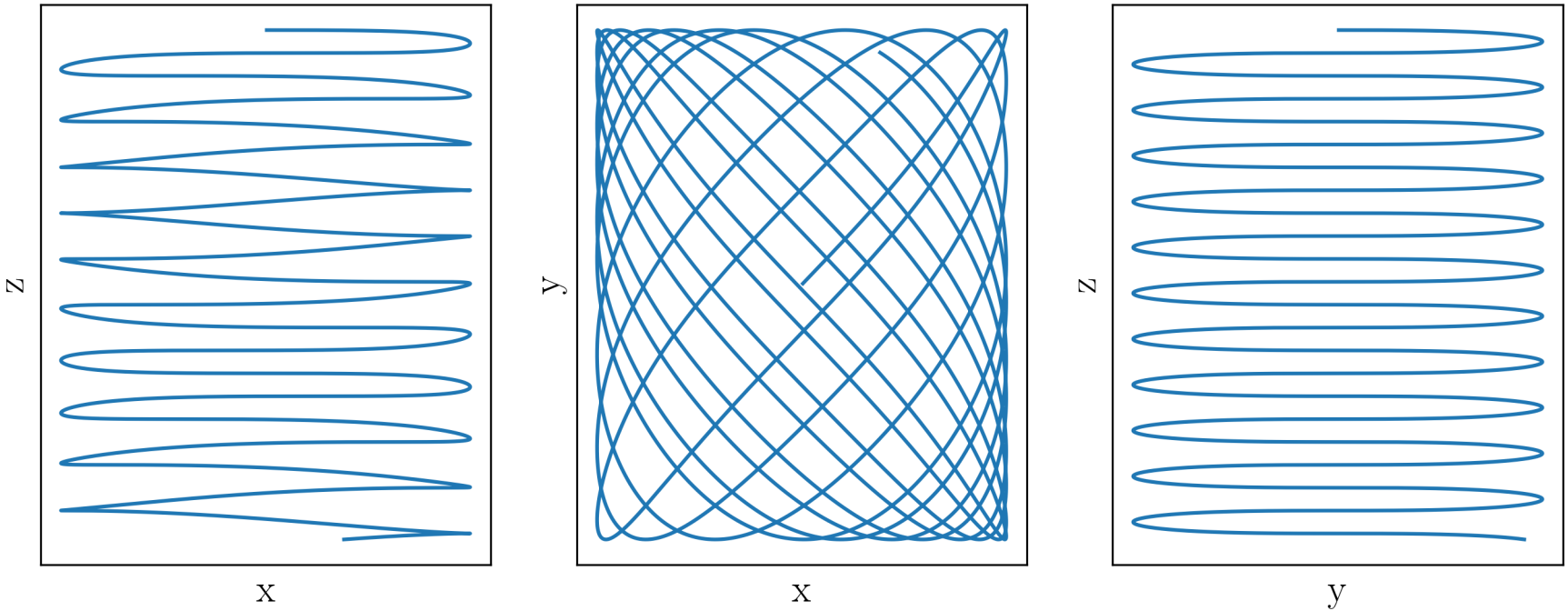

$B_z = 1$

$E_y = B_z - 10^{-3}$

$\gamma_0 = 1$

Boris Boris-tg Vay HC LM VD1 Umeda

# Bounded motion in a spatially inhomogeneous field




$E_x = -x$

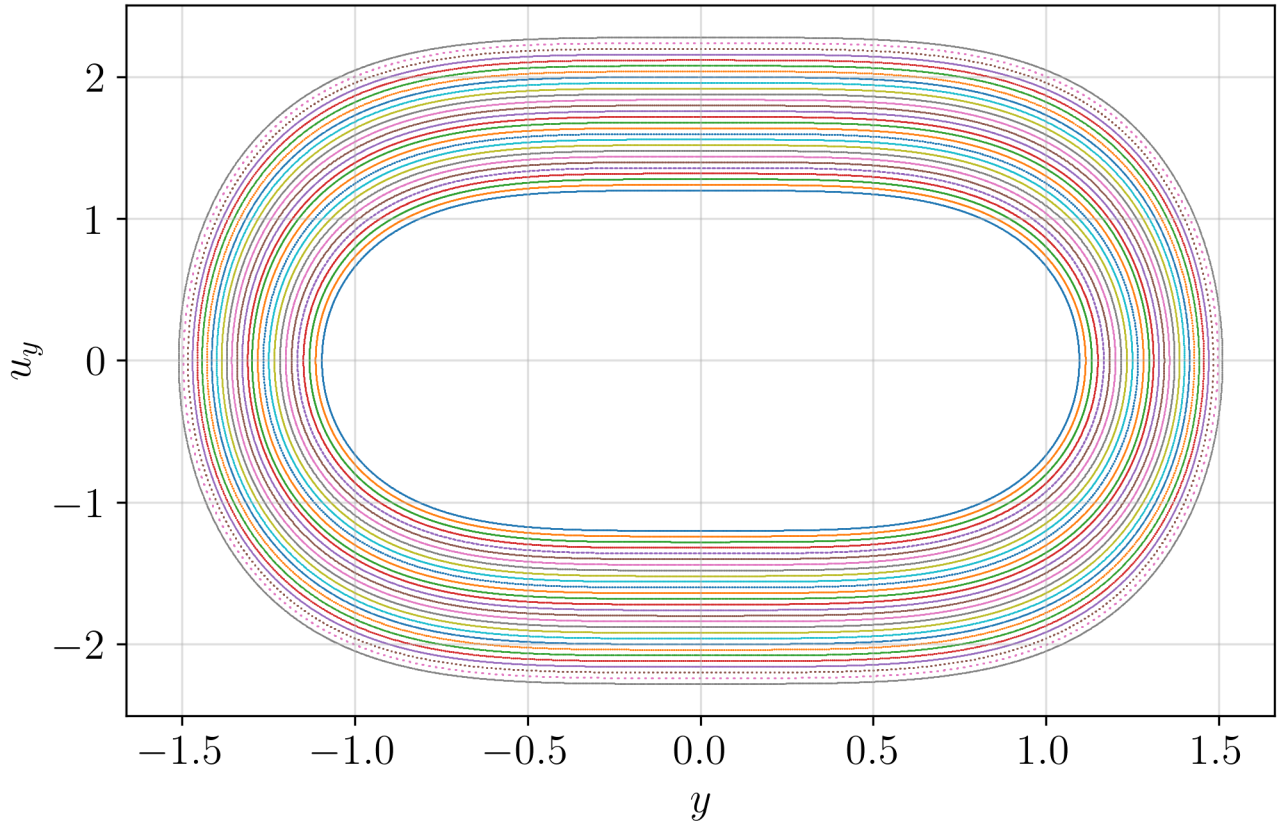
$B_x = 2y$

$\gamma_0 = 4$

Let's plot the Poincaré map $(y, u_y)$, when $x = 0$ and $u_x < 0$.

A. V. Higuera and J. R. Cary. // Physics of plasmas. – 2017. – Vol. 24. – P. 052104.

# Bounded motion in a spatially inhomogeneous field




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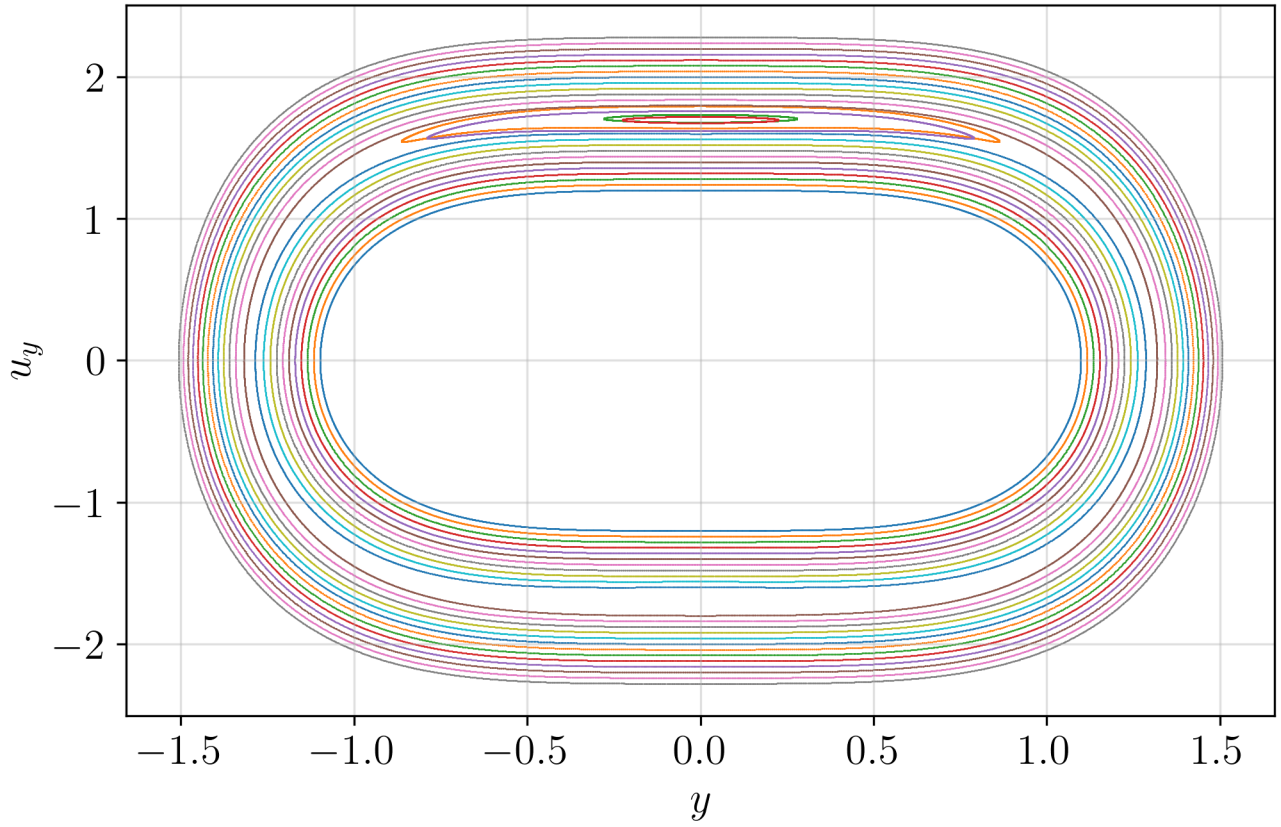
$\gamma_0 = 4$

$u_{y,0} = [1.5, 2.3]$

$u_{z,0} = 0$

$\Delta t = T_{osc}/480$

# Bounded motion in a spatially inhomogeneous field




$E_x = -x$

$B_x = 2\,y$

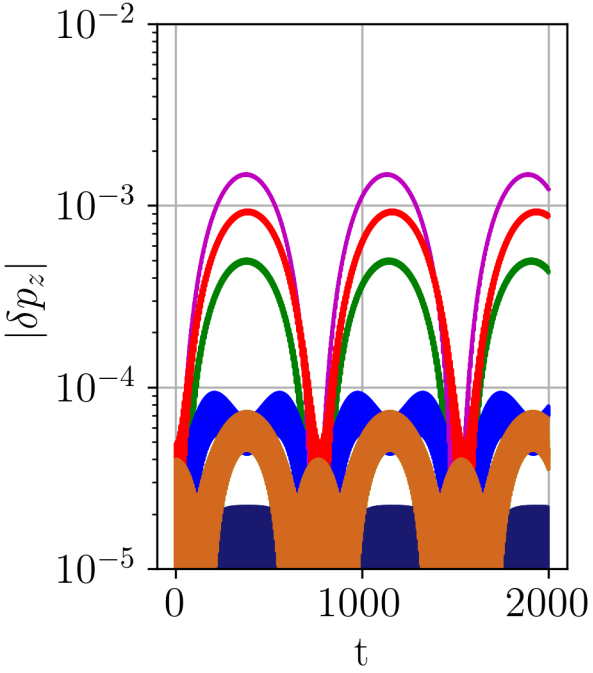
$\gamma_0 = 4$

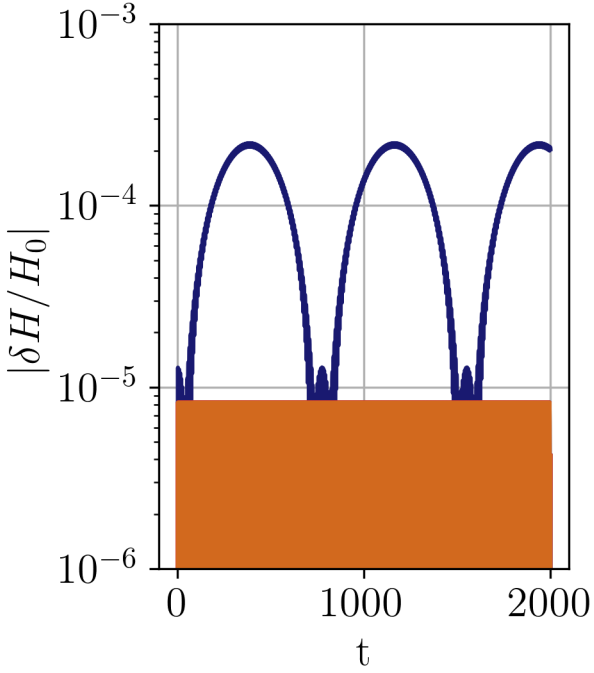
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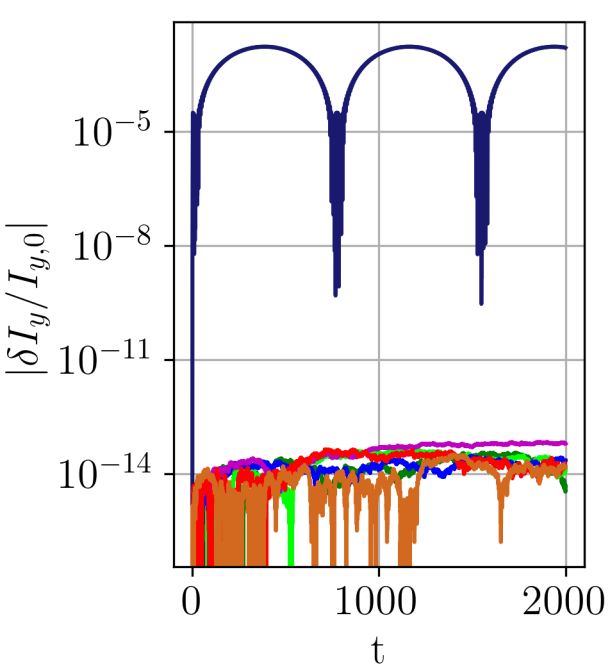
$u_{z,0} = 0$

$\Delta t = T_{osc}/60$

# Bounded motion in a spatially inhomogeneous field




$p_z = const$

$H = \sqrt{1 + p_x^2 + p_y^2 + (p_z - A_z(y))^2} + \phi(x) = const$

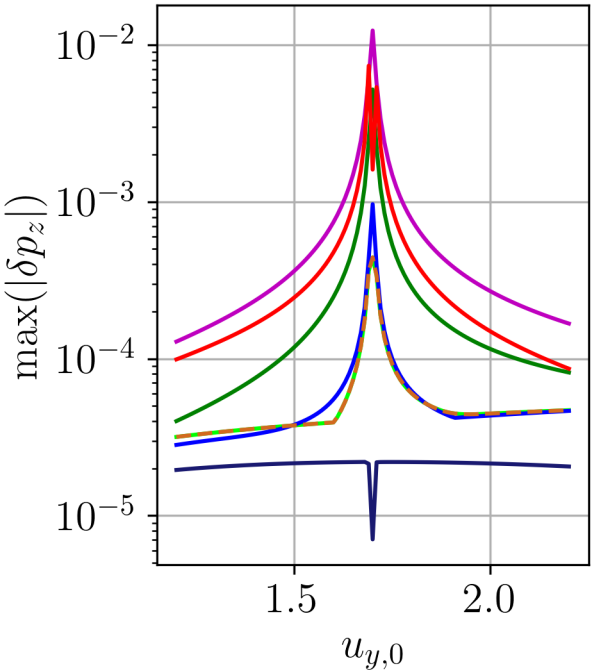
$I_y \equiv p_y^2 + (p_z - A_z(y))^2 = const$

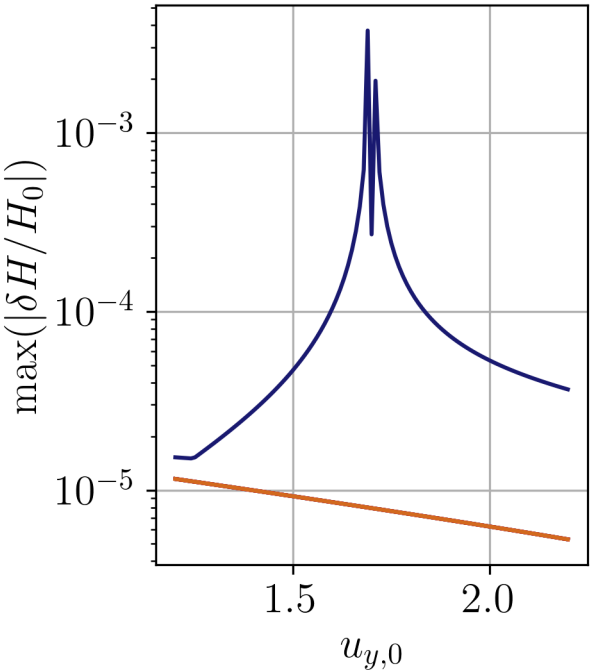
$u_{y,0} = 1.65$

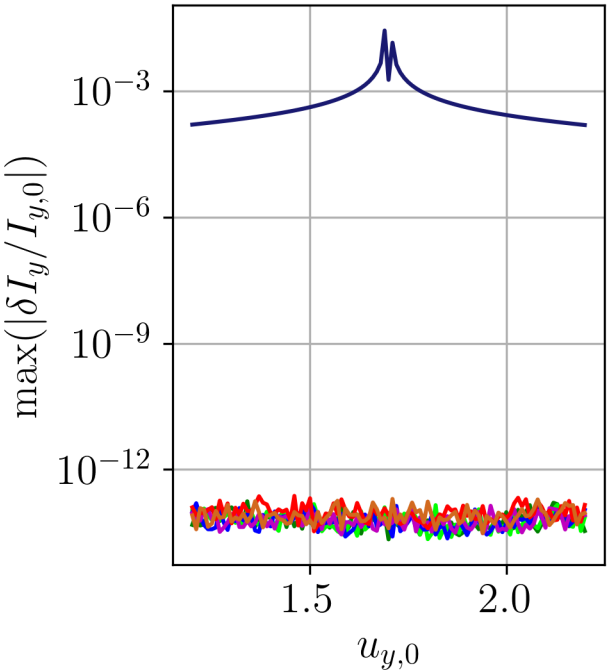
$\Delta t = T_{osc}/480$

Boris Boris-tg Vay HC LM VD1 Umeda

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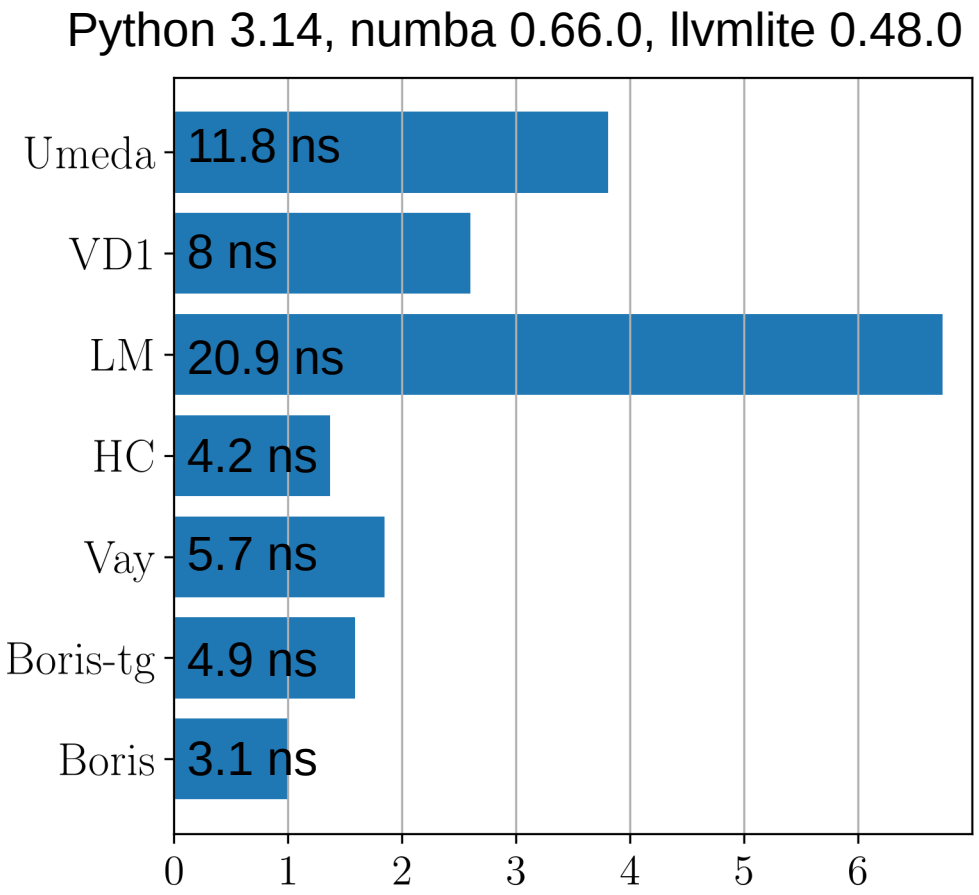
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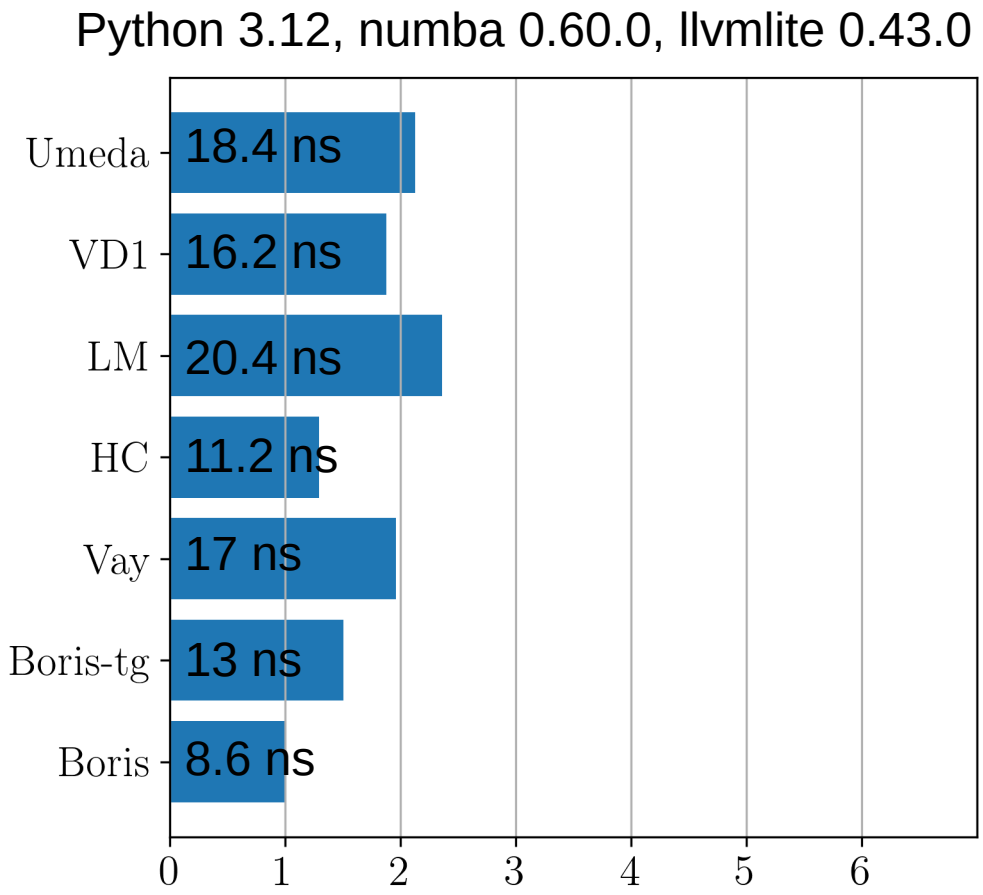
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$\Delta t = T_{osc}/480$

Boris | Boris-tg | Vay | HC | LM | VD1 | Umeda

# Performance




The time is normalized to the time step of the Boris algorithm.

## Conclusion:

- Modern schemes remain highly accurate at large time steps.
- An appropriate time step allows the underlying physical processes to be reproduced.
- Convergence testing is essential.

Thank you!